Induced Norm from L_2 to L_∞ in SISO Sampled-Data Systems

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***Abstract*— This paper investigates the maximum amplitude (i.e., the L_∞ norm) of the output for the worst input with a unit energy (i.e., a unit L_2 norm) in single-input/single-output (SISO) linear time-invariant (LTI) sampled-data systems, by which we mean the generalized plant and the controller are both LTI. It is known that the induced norm from L_2 to L_∞ coincides with the H_2 norm in SISO LTI systems. To highlight the arguments tailored to (SISO) sampled-data systems in this paper, we first see how this induced norm reduces to H_2 norms in the continuous-time and discrete-time cases. Through the lifting-based arguments, we next give a closed-form representation of the induced norm from L_2 to L_∞ in SISO LTI sampled-data systems. We further exploit the associated arguments to compare this induced norm with two existing definitions of the H_2 norm for sampled-data systems, and show that the induced norm coincides with neither of them in SISO LTI sampled-data systems. We further develop a more sophisticated closed-form representation for the induced norm and give an approximate but asymptotically exact method for its computation.**

I. Introduction

The L_2 norm can be used for evaluating the energy of signals, and the L_2-induced norm of a continuous-time LTI system corresponds to the H_∞ norm of the system. Hence, the study associated with the treatment of the L_2-induced norm has been called the H_∞ problem. Furthermore, there have been a number of studies on the continuous-time or discrete-time H_∞ problem [1]–[7] since this system norm has been used as a typical measure in the sensitivity reduction problem and robust control problem.

The L_∞ norm can be used for considering the maximum amplitude of signals, and the L_∞-induced norm of a continuous-time LTI system corresponds to the L_1 norm of the impulse response of the continuous-time system. Thus, the study associated with the treatment of the L_∞-induced norm has been named the L_1 problem. There have been a number of studies on the L_1 problem [8]–[14] because evaluating the maximum amplitude of the output is very important in practice and this problem is pertinent to bounded persistent disturbances often encountered in control systems.

On the other hand, even when the performance analysis for decaying disturbances such as those in L_2 is considered, evaluating the maximum amplitude of the output rather than its L_2 norm may equally play an important role. In other words, computing the induced norm from L_2 to L_∞ could play very important roles in control system analysis. This is indeed true particularly because this induced norm admits an alternative interpretation as the H_2 norm in the single-input/single-output (SISO) LTI case, both for continuous-time and discrete-time systems, [15]–[18], even though the H_2 norm (of a multi-input/multi-output LTI system) is usually defined in the frequency domain and related to the power of the output for a white noise input. Another well-known interpretation of the H_2 norm is related to the L_2 norm of the output for impulse disturbances.

In view of such relevant studies, one could naturally raise a question whether or not the induced norm from L_2 to L_∞ in SISO sampled-data systems coincides with either of the two (conceptually) different definitions for the H_2 norm of LTI sampled-data systems [19]–[22]. The induced norm from L_2 to L_∞ in sampled-data systems was analytically formulated first in [23] by using the idea of the lifting technique [24]–[26], but no explicit computation method for the induced norm was provided in that study. This is partly because the treatment of the induced norm in that study involves an infinite summation, whose explicit computation was not discussed. More importantly, the study is interested in the synthesis of the optimal controller minimizing the induced norm and thus it does not give an exact characterization for analyzing the normAn explicit computation method for the induced norm without the lifting arguments was developed in [27]. However, its comparison with two definitions for the H_2 norm of sampled-data systems was not discussed there. This paper employs the lifting arguments and deals with the induced norm from L_2 to L_∞ in SISO LTI sampled-data systems directly, in such a way that the comparison of the induced norm with two existing definitions for the H_2 norm of sampled-data systems is easy. As it turns out, the arguments in this paper give a negative answer to the aforementioned question on their mutual relation and thus could be interpreted as giving yet another definition of the H_2 norm of SISO LTI sampled-data systems. These discrepancies of the present norm from the existing H_2 norms can be regarded as stemming from another aspect of the hybrid continuous-time/discrete-time nature of sampled-data systems, even though such a nature has already been studied intensively in [20]–[22] in the context of extending the H_2 problems to sampled-data systems.

In the following, we use the notations $\mathbb{N}$ and $\mathbb{R}^\nu$ to denote the set of positive integers and the set of ν-dimensional real vectors, respectively. We further use the notation $\mathbb{N}_0$ to imply $\mathbb{N} \cup \{0\}$. The notation $\|\cdot\|_\infty$ is used to mean either the L_∞ norm of a function, i.e.,

$$\|f(\cdot)\|_\infty := \operatorname*{ess\,sup}_{0\le t<\infty} |f(t)| \tag{1}$$

or the l_∞ norm of a sequence, i.e.,

$$\|g(\cdot)\|_\infty := \sup_{k\in\mathbb{N}_0} |g(k)| \tag{2}$$

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or the ∞-norm of a finite-dimensional matrix (induced from the vector ∞-norm), whose distinction will be clear from the context. On the other hand, the notation $\|\cdot\|_2$ is used to mean either the L_2 norm of a function, i.e.,

$$\|f(\cdot)\|_2 := \left(\int_0^\infty f^2(t)dt\right)^{1/2} \tag{3}$$

or the l_2 norm of a sequence, i.e.,

$$\|g(\cdot)\|_2 := \left(\sum_{k=0}^\infty g^2(k)\right)^{1/2} \tag{4}$$

or the 2-norm of a finite-dimensional matrix (induced from the vector Euclidean norm), whose distinction will be also clear from the context. Let $\mathbf{T}$ be an operator from L_2 to L_∞ (or from l_2 to l_∞). Then, the notation $\|\cdot\|_{\infty/2}$ is used to denote the induced norm from L_2 to L_∞ (or from l_2 to l_∞), i.e.,

$$\|\mathbf{T}\|_{\infty/2} := \sup_{\|w\|_2\le 1} \|\mathbf{T}w\|_\infty \tag{5}$$

Furthermore, we call the induced norms from L_2 to L_∞ and from l_2 to l_∞ the L_∞/L_2-induced norm and l_∞/l_2-induced norm, respectively, for simplicity.

II. Continuous-Time and Discrete-Time Cases

As mentioned in the preceding section, the L_∞/L_2-induced norm of SISO continuous-time LTI systems and the l_∞/l_2-induced norm of SISO discrete-time LTI systems have been known to coincide with the continuous-time H_2 norm and the discrete-time H_2 norm, respectively. However, this fact has been stated only without proof in [8] and [28], while [15]–[18] deal also with relevant topics and explicit proofs are given only for such topics. Recovering the proof of the above fact from the relevant proofs may not necessarily be extremely hard but not straightforward, either, while giving an explicit proof is expected to be helpful in highlighting the arguments of the present paper tailored to (SISO) sampled-data systems. Hence, this section is devoted to such an explicit proof.

A. Continuous-Time Case

We first consider the continuous-time case. Let us consider the stable continuous-time SISO FDLTI system

$$G_\mathrm{c}: \begin{cases} \dfrac{dx}{dt} & = A_\mathrm{c}x + B_\mathrm{c}w \\ z & = C_\mathrm{c}x \end{cases} \tag{6}$$

where $x(t) \in \mathbb{R}^n$ is the state, $w(t) \in \mathbb{R}$ is the input and $z(t) \in \mathbb{R}$ is the output. Throughout the paper, we assume that $x(0) = 0$. Then,

$$\begin{aligned} z(t) &= \int_0^t C_\mathrm{c}\exp(A_\mathrm{c}(t-\tau))B_\mathrm{c}w(\tau)d\tau \\ &=: (\mathbf{T}_\mathrm{c}w)(t) \quad (0 \le t < \infty) \end{aligned} \tag{7}$$

$\mathbf{T}_\mathrm{c}$ defined above for the system (6) is known to be an operator from L_2 to L_∞. By noting that $\mathbf{T}_\mathrm{c}w$ is a continuous function and the system (6) is LTI, it is easy to see that the L_∞/L_2-induced norm of the system (6) can be described by

$$\begin{aligned} \|\mathbf{T}_\mathrm{c}\|_{\infty/2} &:= \sup_{\|w\|_2\le 1}\|\mathbf{T}_\mathrm{c}w\|_\infty \\ &= \sup_{\|w\|_2\le 1}\sup_t |(\mathbf{T}_\mathrm{c}w)(t)| = \sup_t\sup_{\|w\|_2\le 1}|(\mathbf{T}_\mathrm{c}w)(t)| \\ &= \sup_t\sup_{\|w\|_2\le 1}\left|\int_0^t C_\mathrm{c}\exp(A_\mathrm{c}(t-\tau))B_\mathrm{c}w(\tau)d\tau\right| \\ &= \sup_t\sup_{\|w\|_2\le 1}\left|\int_0^t C_\mathrm{c}\exp(A_\mathrm{c}\theta)B_\mathrm{c}w(t-\theta)d\theta\right| \\ &=: \sup_t\sup_{\|u\|_2\le 1}\left|\int_0^t C_\mathrm{c}\exp(A_\mathrm{c}\theta)B_\mathrm{c}u(\theta)d\theta\right| \\ &=: \sup_t\sup_{\|u\|_2\le 1}|(\mathbf{F}_\mathrm{c}u)(t)|_\infty = \lim_{t\to\infty}\sup_{\|u\|_2\le 1}|(\mathbf{F}_\mathrm{c}u)(t)| \\ &=: \|\mathbf{F}_\mathrm{c}\|_{\infty/2} \end{aligned} \tag{8}$$

Remark 1: $\mathbf{F}_c$ defined above is also regarded as an operator from L_2 to L_∞. In the following, we compute the L_∞/L_2-induced norm $\|\mathbf{T}_\mathrm{c}\|_{\infty/2}$ by computing the L_∞/L_2-induced norm $\|\mathbf{F}_\mathrm{c}\|_{\infty/2}$ instead because of some simplicities in the following arguments.

Here, we review the continuous-time Cauchy-Schwarz inequality, with the functions f and g, given by

$$\left(\int_0^t f(\theta)g(\theta)d\theta\right)^2 \le \int_0^t f^2(\theta)d\theta\cdot\int_0^t g^2(\theta)d\theta \tag{9}$$

where the equality holds if and only if $f = \lambda g$ on $[0, t]$ for a constant λ. By this inequality, we have

$$\begin{aligned} \|\mathbf{F}_\mathrm{c}\|_{\infty/2} &= \lim_{t\to\infty}\sup_{\|u\|_2\le 1}\left|\int_0^t C_\mathrm{c}\exp(A_\mathrm{c}\theta)B_\mathrm{c}u(\theta)d\theta\right| \\ &= \left(\int_0^\infty C_\mathrm{c}\exp(A_\mathrm{c}\theta)B_\mathrm{c}B_\mathrm{c}^T\exp(A_\mathrm{c}^T\theta)C_\mathrm{c}^T d\theta\right)^{1/2} \end{aligned} \tag{10}$$

because

$$\begin{aligned} &\left(\int_0^t C_\mathrm{c}\exp(A_\mathrm{c}\theta)B_\mathrm{c}u(\theta)d\theta\right)^2 \\ &\le \int_0^t C_\mathrm{c}\exp(A_\mathrm{c}\theta)B_\mathrm{c}B_\mathrm{c}^T\exp(A_\mathrm{c}^T\theta)C_\mathrm{c}^T d\theta\cdot\int_0^t u^2(\theta)d\theta \end{aligned} \tag{11}$$

Thus, by the Plancherel theorem, we can see from (10) that the L_∞/L_2-induced norm $\|\mathbf{F}_\mathrm{c}\|_{\infty/2}$ coincides with the H_2 norm associated with (the transfer function of) the SISO continuous-time LTI system (6).

B. Discrete-Time Case

We next consider the discrete-time case. Let us consider the stable discrete-time SISO FDLTI system

$$G_\mathrm{d}: \begin{cases} x(k+1) & = A_\mathrm{d}x(k) + B_\mathrm{d}w(k) \\ z(k) & = C_\mathrm{d}x(k) + D_\mathrm{d}w(k) \end{cases} \tag{12}$$

where $x(k) \in \mathbb{R}^n$ is the state, $w(k) \in \mathbb{R}$ is the input and $z(k) \in \mathbb{R}$ is the output. Assuming that $x(0) = 0$,

$$\begin{aligned} z(k) &= \sum_{i=0}^{k-1} C_\mathrm{d}A_\mathrm{d}^i B_\mathrm{d}w(k-1-i) + D_\mathrm{d}w(k) \\ &=: (\mathbf{T}_\mathrm{d}w)(k) \quad (k \in \mathbb{N}_0) \end{aligned} \tag{13}$$

$\mathbf{T}_{\rm d}$ defined above is known to be an operator from l_2 to l_∞. Similarly to the continuous-time case, the l_∞/l_2-induced norm of the system (12) can be given by

$$\begin{aligned}
\|\mathbf{T}_{\rm d}\|_{\infty/2} &= \sup_{\|w\|_2\le 1} \|\mathbf{T}_{\rm d}w\|_\infty \\
&= \sup_{\|w\|_2\le 1}\sup_k |(\mathbf{T}_{\rm d}w)(k)| = \sup_k \sup_{\|w\|_2\le 1} |(\mathbf{T}_{\rm d}w)(k)| \\
&= \sup_k \sup_{\|w\|_2\le 1} \left| \sum_{i=0}^{k-1} C_{\rm d}A_{\rm d}^i B_{\rm d} w(k-1-i) + D_{\rm d}w(k)\right| \\
&=: \sup_k \sup_{\|u\|_2\le 1} \left| \sum_{i=1}^{k} C_{\rm d}A_{\rm d}^{i-1} B_{\rm d} u(i) + D_{\rm d}u(0)\right| \\
&=: \sup_k \sup_{\|u\|_2\le 1} |(\mathbf{F}_{\rm d}u)(k)| = \lim_{k\to\infty} \sup_{\|u\|_2\le 1} |(\mathbf{F}_{\rm d}u)(k)| \\
&=: \|\mathbf{F}_{\rm d}\|_{\infty/2}
\end{aligned} \tag{14}$$

Remark 2: Similarly to the continuous-time case, we compute the l_∞/l_2-induced norm $\|\mathbf{T}_{\rm d}\|_{\infty/2}$ by computing the l_∞/l_2-induced norm $\|\mathbf{F}_{\rm d}\|_{\infty/2}$ instead.

Now, the discrete-time Cauchy-Schwarz inequality, with the sequences $f(i)$ and $g(i)$, states that

$$\left(\sum_{i=0}^{k} f(i)g(i)\right)^2 \le \sum_{i=0}^{k} f^2(i)\cdot \sum_{i=0}^{k} g^2(i) \tag{15}$$

where the equality holds if and only if $f(i) = \lambda g(i),\ i = 0,\ldots,k$ for a constant λ. Hence, it readily follows that

$$\begin{aligned}
\|\mathbf{F}_{\rm d}\|_{\infty/2} &= \lim_{k\to\infty} \sup_{\|u\|_2\le 1} \left| \sum_{i=1}^{k} C_{\rm d}A_{\rm d}^{i-1} B_{\rm d} u(i) + D_{\rm d}u(0)\right| \\
&= \left(\sum_{i=0}^{\infty} C_{\rm d}A_{\rm d}^i B_{\rm d} B_{\rm d}^T (A_{\rm d}^T)^i C_{\rm d}^T + D_{\rm d}D_{\rm d}^T\right)^{1/2}
\end{aligned} \tag{16}$$

because

$$\begin{aligned}
&\left(\sum_{i=1}^{k} C_{\rm d}A_{\rm d}^i B_{\rm d} u(i) + D_{\rm d}u(0)\right)^2 \\
\le &\left(\sum_{i=1}^{k} C_{\rm d}A_{\rm d}^i B_{\rm d} B_{\rm d}^T (A_{\rm d}^T)^i C_{\rm d}^T + D_{\rm d}D_{\rm d}^T\right)\cdot \sum_{i=0}^{k} u^2(i)
\end{aligned} \tag{17}$$

Thus, we can see from (16) that the l_∞/l_2-induced norm $\|\mathbf{F}_{\rm d}\|_{\infty/2}$ coincides with the H_2 norm associated with (the transfer function of) the SISO discrete-time LTI system (12).

III. L_∞/L_2-Induced Norm of SISO Sampled-Data Systems

In the preceding section, we gave an explicit proof of the fact that the L_∞/L_2-induced norm $\|\mathbf{F}_{\rm c}\|_{\infty/2}$ and the l_∞/l_2 induced norm $\|\mathbf{F}_{\rm d}\|_{\infty/2}$ coincide with the continuous-time and discrete-time H_2 norms, respectively, where the main idea was the application of the Cauchy-Schwarz inequalities. Since these induced norms of SISO continuous-time and discrete-time LTI systems coincides with the continuous-time and discrete-time H_2 norms, respectively, it is interesting to ask whether the same is true for SISO LTI sampled-data systems; more precisely, since there are two different definitions for the H_2 norm of LTI sampled-data systems [19]–[22], whether the L_∞/L_2-induced norm coincides with either of the two definitions. In this regard, it is also of interest to see whether or not the Cauchy-Schwarz inequalities can be directly applied to the sampled-data case. This section is devoted to such arguments and gives a negative answer to the question on the relationship with the H_2 norm.

A. L_∞/L_2-Induced Norm and Its Lifting-Based Treatment

Let us consider the stable sampled-data system $\Sigma_{\rm SD}$ shown in Fig. 1, where P denotes the continuous-time LTI generalized plant, while Ψ, $\mathcal{H}$ and $\mathcal{S}$ denote the discrete-time LTI controller, the zero-order hold and the ideal sampler, respectively, operating with sampling period h in a synchronous fashion. Solid lines and dashed lines in Fig. 1 are used to represent continuous-time signals and discrete-time signals, respectively. Suppose that P and Ψ are described respectively by

$$P:\begin{cases} \dfrac{dx}{dt} = Ax + B_1 w + B_2 u \\ z = C_1 x + D_{12} u \\ y = C_2 x \end{cases} \tag{18}$$

$$\Psi:\begin{cases} \psi_{k+1} = A_\Psi \psi_k + B_\Psi y_k \\ u_k = C_\Psi \psi_k + D_\Psi y_k \end{cases} \tag{19}$$

where $x(t)\in\mathbb{R}^n,\ w(t)\in\mathbb{R},\ u(t)\in\mathbb{R}^{n_u},\ z(t)\in\mathbb{R},\ y(t)\in\mathbb{R}^{n_y},\ \psi_k\in\mathbb{R}^{n_\Psi},\ y_k = y(kh)$ and $u(t) = u_k\ (kh\le t<(k+1)h)$. Note that we have assumed '$D_{11}=0$' and '$D_{21}=0$' in the description (18) of the continuous-time generalized plant P. This is necessary (and sufficient by the stability of $\Sigma_{\rm SD}$) for the L_∞/L_2-induced norm $\sup\limits_{\|w\|_2\le 1}\|z\|_\infty$ of the sampled-data system $\Sigma_{\rm SD}$ to be bounded/well-defined.

Because the sampled-data system $\Sigma_{\rm SD}$ is a hybrid continuous-time/discrete-time system, this system viewed in continuous-time is (periodically) time-varying. To deal with $\Sigma_{\rm SD}$ as a time-invariant system, we apply the lifting technique [24]–[26]. That is, given $f\in L_\infty$ or $f\in L_2$, its lifting $\{\widehat{f}_k\}_{k=0}^\infty$ with $\widehat{f}_k\in L_\infty[0,h)$ or $L_2[0,h)$ $(k\in\mathbb{N}_0)$ (with sampling period h) is defined as follows [24]–[26]:

$$\widehat{f}_k(\theta) = f(kh+\theta) \quad (0\le\theta<h) \tag{20}$$

By applying lifting to $w\in L_2$ and $z\in L_\infty$, the lifted representation of the sampled-data system $\Sigma_{\rm SD}$ is described by

$$\begin{cases} \xi_{k+1} &= \mathcal{A}\xi_k + \mathcal{B}\widehat{w}_k \\ \widehat{z}_k &= \mathcal{C}\xi_k + \mathcal{D}\widehat{w}_k \end{cases} \tag{21}$$

with $\xi_k := [x_k^T\ \ \psi_k^T]^T\ (x_k := x(kh))$, the matrix

$$\mathcal{A} = \begin{bmatrix} A_d + B_{2d}D_\Psi C_{2d} & B_{2d}C_\Psi \\ B_\psi C_{2d} & A_\Psi \end{bmatrix} : \mathbb{R}^{n+n_\Psi}\to\mathbb{R}^{n+n_\Psi} \tag{22}$$

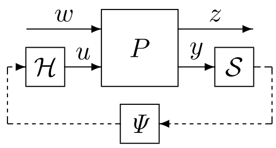

Fig. 1. Sampled-data system $\Sigma_{\rm SD}$.

and the operators

$$\mathcal{B} = J\mathbf{B}_1 : L_2[0,h) \to \mathbb{R}^{n+n_\Psi} \tag{23}$$
$$\mathcal{C} = \mathbf{M}_1 C_\Sigma : \mathbb{R}^{n+n_\Psi} \to L_\infty[0,h) \tag{24}$$
$$\mathcal{D} = \mathbf{D}_{11} : L_2[0,h) \to L_\infty[0,h) \tag{25}$$

where

$$A_d := \exp(Ah),\ B_{2d} := \int_0^h \exp(A\theta)B_2 d\theta,\ C_{2d} := C_2 \tag{26}$$

$$J := \begin{bmatrix} I \\ 0 \end{bmatrix} \in \mathbb{R}^{(n+n_\Psi)\times n},\ C_\Sigma := \begin{bmatrix} I & 0 \\ D_\Psi C_{2d} & C_\Psi \end{bmatrix} \tag{27}$$

$$\mathbf{B}_1 w = \int_0^h \exp(A(h-\theta))B_1 w(\theta) d\theta \tag{28}$$

$$\left(\mathbf{M}_1 \begin{bmatrix} x \\ u \end{bmatrix}\right)(\theta) = C_0 \exp(A_2\theta) \begin{bmatrix} x \\ u \end{bmatrix} \tag{29}$$

$$A_2 := \begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix},\ C_0 := \begin{bmatrix} C_1 & D_{12} \end{bmatrix} \tag{30}$$

$$(\mathbf{D}_{11}w)(\theta) = \int_0^\theta C_1 \exp(A(\theta-\tau))B_1 w(\tau) d\tau \tag{31}$$

From the stability assumption of $\Sigma_{\rm SD}$, $\mathcal{A}$ is stable, i.e., has all its eigenvalues in the open unit disc.

Once the discrete-time LTI representation (21) of the sampled-data system $\Sigma_{\rm SD}$ is obtained by applying lifting (for simplicity, we say the sampled-data system $\Sigma_{\rm SD}$ is LTI for the existence of such a representation), one may consider that its L_∞/L_2-induced norm could be easily computed through some technique similar to that employed in the computation of the l_∞/l_2-induced norm of discrete-time systems. However, (21) is actually quite different from the state equation (12) for discrete-time systems because $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are operators. Hence, the discrete-time Cauchy-Schwarz inequality, which is the key technique in showing the equivalence of the l_∞/l_2-induced norm and the H_2 norm in SISO discrete-time LTI systems, cannot be directly applied to the lifted representation (21). Consequently, we need to develop a method specific to (SISO) sampled-data systems. The details of the numerical computation method will be discussed in Section IV, and we restrict our attention in this section to a possible relationship of the L_∞/L_2 induced norm with the two existing definitions of the H_2 norm of sampled-data systems. More specifically, this subsection is devoted to a preliminary consideration, which is then exploited in the following subsection to study such a possible relationship in more detail.

To give an alternative characterization of the L_∞/L_2-induced norm of the SISO LTI sampled-data system $\Sigma_{\rm SD}$ in the lifting-based framework, we first note (21) and describe the closed-loop relation between $\widehat{w}_k$ and $\widehat{z}_k$ $(k = 0, \cdots, \infty)$ as follows:

$$\begin{bmatrix} \widehat{z}_0 \\ \widehat{z}_1 \\ \widehat{z}_2 \\ \widehat{z}_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} \mathcal{D} & 0 & \cdots & & & \\ \mathcal{CB} & \mathcal{D} & 0 & \cdots & & \\ \mathcal{CAB} & \mathcal{CB} & \mathcal{D} & 0 & \cdots & \\ \mathcal{CA}^2\mathcal{B} & \mathcal{CAB} & \mathcal{CB} & \mathcal{D} & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} \widehat{w}_0 \\ \widehat{w}_1 \\ \widehat{w}_2 \\ \widehat{w}_3 \\ \vdots \end{bmatrix} \tag{32}$$

Because the lifting is norm-preserving in both L_∞ and L_2, the L_∞/L_2-induced norm of the sampled-data system $\Sigma_{\rm SD}$ coincides with the L_∞/L_2-induced norm of the above operator in the right hand side. Furthermore, since this operator has a Toeplitz structure (and thus every row is an extension of the previous row), it readily follows that the L_∞/L_2-induced norm of $\Sigma_{\rm SD}$ coincides with the L_∞/L_2-induced norm of its "last" block row, i.e., (after reordering without affecting the L_∞/L_2-induced norm)

$$\mathcal{F} := \begin{bmatrix} \mathcal{D} & \mathcal{CB} & \mathcal{CAB} & \mathcal{CA}^2\mathcal{B} & \cdots \end{bmatrix} \tag{33}$$

The L_∞/L_2-induced norm $\|\mathcal{F}\|_{\infty/2}$ is defined as

$$\begin{aligned} \|\mathcal{F}\|_{\infty/2} &:= \sup_{\|\widehat{w}\|_2 \le 1} \|(\mathcal{F}\widehat{w})(\cdot)\|_\infty \\ &= \sup_{\|\widehat{w}\|_2 \le 1} \sup_{0\le\theta<h} |(\mathcal{F}\widehat{w})(\theta)| = \sup_{0\le\theta<h} \sup_{\|\widehat{w}\|_2 \le 1} |(\mathcal{F}\widehat{w})(\theta)| \end{aligned} \tag{34}$$

where $\widehat{w} =: [\widehat{w}_0,\ \widehat{w}_1, \cdots]^T$. For a fixed $\theta \in [0,h)$, we have

$$\begin{aligned} (\mathcal{F}\widehat{w})(\theta) &= (\mathcal{D}\widehat{w}_0)(\theta) + (\mathcal{CB}\widehat{w}_1)(\theta) + (\mathcal{CAB}\widehat{w}_2)(\theta) + \cdots \\ &= \int_0^\theta D_\theta(\tau)\widehat{w}_0(\tau)d\tau + \int_0^h C_\theta B_h(\tau)\widehat{w}_1(\tau)d\tau \\ &\quad + \int_0^h C_\theta \mathcal{A} B_h(\tau)\widehat{w}_2(\tau)d\tau + \cdots \end{aligned} \tag{35}$$

with the matrix functions

$$B_h(\tau) = J\exp(A(h-\tau))B_1 \tag{36}$$
$$D_\theta(\tau) = C_1 \exp(A(\theta-\tau))B_1 \tag{37}$$

and the matrix $C_\theta = C_0 \exp(A_2\theta)C_\Sigma$. Applying the continuous-time Cauchy-Schwarz inequality to (35) leads to

$$\begin{aligned} |(\mathcal{F}\widehat{w})(\theta)| &\le \left(\int_0^\theta D_\theta^2(\tau)d\tau\right)^{1/2} \cdot \left(\int_0^\theta \widehat{w}_0^2(\tau)d\tau\right)^{1/2} \\ &\quad + \left(\int_0^h (C_\theta B_h(\tau))^2 d\tau\right)^{1/2} \cdot \left(\int_0^h \widehat{w}_1^2(\tau)d\tau\right)^{1/2} + \cdots \end{aligned} \tag{38}$$

Furthermore, by applying the discrete-time Cauchy-Schwarz inequality to (38), it easily follows that

$$\begin{aligned} &\sup_{\|\widehat{w}\|_2 \le 1} |(\mathcal{F}\widehat{w})(\theta)| \\ &\le \left(\int_0^\theta D_\theta^2(\tau)d\tau + \int_0^h (C_\theta B_h(\tau))^2 d\tau + \int_0^h (C_\theta \mathcal{A} B_h(\tau))^2 d\tau \right. \\ &\qquad \left. + \int_0^h (C_\theta \mathcal{A}^2 B_h(\tau))^2 d\tau + \cdots \right)^{1/2} =: F(\theta) \end{aligned} \tag{39}$$

Remark 3: The infinite series (39) is convergent by the stability assumption of $\mathcal{A}$.

In particular, if we construct $\widehat{w}$ as

$$\widehat{w}_0(\tau) = \begin{cases} \lambda D_\theta(\tau) & (0 \le \tau < \theta) \\ 0 & (\theta \le \tau < h) \end{cases} \tag{40}$$

$$\widehat{w}_i(\tau) = \lambda C_\theta \mathcal{A}^i B_h(\tau) \quad (0 \le \tau < h,\ i \in \mathbb{N}) \tag{41}$$

where $\lambda := 1/F(\theta)$, we then easily see that $\|\widehat{w}\|_2 = 1$ and the equalities hold both in (38) and (39). This immediately implies that $\sup_{\|\widehat{w}\|_2 \le 1} |(\mathcal{F}\widehat{w})(\theta)| = F(\theta)$. Thus, by (34), the L_∞/L_2-induced norm $\|\mathcal{F}\|_{\infty/2}$ can be given by

$$\|\mathcal{F}\|_{\infty/2} = \sup_{0 \le \theta < h} F(\theta) \tag{42}$$

B. Relationship between the L_∞/L_2-Induced Norm and Existing Definitions of the H_2 Norm of Sampled-Data Systems

Based on the alternative characterization (42) of the L_∞/L_2-induced norm of SISO LTI sampled-data systems, this subsection is devoted to discussing whether this induced norm coincides with either of the two existing definitions of the H_2 norm of sampled-data systems. This is a natural question because this induced norm does coincide with the H_2 norm for SISO continuous-time LTI systems (and the l_∞/l_2-induced norm coincides with the H_2 norm for SISO discrete-time LTI systems). We begin by reviewing the two definitions for the H_2 norm of (SISO) LTI sampled-data systems. The first definition [19] considers the L_2 norm of the regulated output $z(t)$ for the impulse input $w(t) = \delta(t)$ occurring at $t = 0$, an instant at which the sampler takes its action. The second definition [20]–[22], on the other hand, considers the root mean square of the L_2 norms of different responses of $z(t)$ for the impulse inputs $w(t)$ occurring at any instants in $[0, h)$. The precise definitions are as follows.

1) H_2 norm definition through a single impulse input [19]: When $w(t) = \delta(t)$, we can formally regard that its lifted representation is given by

$$\begin{cases} \widehat{w}_0 = \delta(\theta) \\ \widehat{w}_i = 0 \quad (i \in \mathbb{N}) \end{cases} \tag{43}$$

By evaluating the L_2 norm of the corresponding output, the H_2 norm of the (SISO) LTI sampled-data system Σ_{SD}, denoted by $\|\Sigma_{\mathrm{SD}}\|_{H_2}^{[0]}$, is defined as

$$\|\Sigma_{\mathrm{SD}}\|_{H_2}^{[0]} := \left\| \begin{bmatrix} \mathcal{D}\delta & \mathcal{C}\mathcal{B}\delta & \mathcal{C}\mathcal{A}\mathcal{B}\delta & \cdots \end{bmatrix}^T \right\|_2 \tag{44}$$

$$= \left(\int_0^h D_h(\theta)^2 d\theta + \int_0^h (C_\theta B_h(0))^2 d\theta + \int_0^h (C_\theta \mathcal{A} B_h(0))^2 d\theta + \int_0^h (C_\theta \mathcal{A}^2 B_h(0))^2 d\theta + \cdots \right)^{1/2} \tag{45}$$

where $\|\cdot\|_2$ in (44) denotes the $L_2[0, h)$ norm of an infinite-dimensional vector function on $[0, h)$, i.e., $\|f\|_2 := (\int_0^h f^T(\theta) f(\theta) d\theta)^{1/2}$. It is easy to see from (39) and (45) that the variables of integration in (39) are different from those in (45), and this is expected to lead to $F(\theta)$ different from $\|\Sigma_{\mathrm{SD}}\|_{H_2}^{[0]}$, for all $\theta \in [0, h)$. This suggests that the L_∞/L_2-induced norm $\|\mathcal{F}\|_{\infty/2}$ is intrinsically different from the H_2 norm $\|\Sigma_{\mathrm{SD}}\|_{H_2}^{[0]}$ in [19].

2) H_2 norm definition through averaging about impulse inputs [20]–[22]: By considering the impulse inputs $w(t) = \delta_\tau(t) := \delta(t - \tau)$ for all $\tau \in [0, h)$, another H_2 norm, denoted by $\|\Sigma_{\mathrm{SD}}\|_{H_2}^{[0,h)}$, is defined as the root mean square of the L_2 norms of $z(t)$ for these impulse inputs as

$$\|\Sigma_{\mathrm{SD}}\|_{H_2}^{[0,h)} := \left(\frac{1}{h} \int_0^h \| [\mathcal{D}\delta_\tau\ \ \mathcal{C}\mathcal{B}\delta_\tau\ \ \mathcal{C}\mathcal{A}\mathcal{B}\delta_\tau\ \ \cdots]^T \|_2^2 d\tau \right)^{1/2}$$

$$= \frac{1}{\sqrt{h}} \left(\int_0^h \int_0^\theta (D_\theta(\tau))^2 d\tau d\theta + \int_0^h \int_0^h (C_\theta B_h(\tau))^2 d\tau d\theta + \int_0^h \int_0^h (C_\theta \mathcal{A} B_h(\tau))^2 d\tau d\theta + \cdots \right)^{1/2} \tag{46}$$

It is very interesting to see from (39) and (46) that

$$\|\Sigma_{\mathrm{SD}}\|_{H_2}^{[0,h)} = \left(\frac{1}{h} \int_0^h F^2(\theta) d\theta \right)^{1/2} \tag{47}$$

while the L_∞/L_2-induced norm $\|\mathcal{F}\|_{\infty/2}$ is described by $\sup_{0 \le \theta < h} F(\theta)$ as shown in (42). Hence,

$$\|\Sigma_{\mathrm{SD}}\|_{H_2}^{[0,h)} \le \|\mathcal{F}\|_{\infty/2} \tag{48}$$

follows immediately and it is suggested that L_∞/L_2-induced norm $\|\mathcal{F}\|_{\infty/2}$ is intrinsically different also from the H_2 norm $\|\Sigma_{\mathrm{SD}}\|_{H_2}^{[0,h)}$.

Summarizing the above arguments, we could conclude that the L_∞/L_2-induced norm $\|\mathcal{F}\|_{\infty/2}$ of SISO LTI sampled-data systems may not be characterized by either of the two H_2 norms of sampled-data systems given so far in [19]–[22].

Remark 4: When we consider SISO continuous-time LTI systems as a special class of sampled-data systems, $F(\theta)$ in (39) becomes a constant function on $[0, h)$.

IV. Computation Method of the L_∞/L_2-Induced Norm in SISO LTI Sampled-Data Systems

This section gives methods for computing $F(\theta)$ in (39) and the L_∞/L_2-induced norm $\|\mathcal{F}\|_{\infty/2} = \sup_{0 \le \theta < h} F(\theta)$. For a fixed $\theta \in [0, h]$, we first consider the controllability Grammian

$$W_\theta := \int_0^\theta \exp(A(\theta - \tau)) B_1 B_1^T \exp(A^T(\theta - \tau)) d\tau \tag{49}$$

Then, it is easy to see that

$$\int_0^\theta D_\theta^2(\tau) d\tau = C_1 W_\theta C_1^T \tag{50}$$

$$\int_0^h (C_\theta \mathcal{A}^i B_h(\tau))^2 d\tau = C_\theta \mathcal{A}^i \begin{bmatrix} W_h & 0 \\ 0 & 0 \end{bmatrix} (\mathcal{A}^T)^i C_\theta^T \quad (i \in \mathbb{N}_0) \tag{51}$$

Hence

$$F^2(\theta) = C_1 W_\theta C_1^T + C_\theta \left(\sum_{i=0}^\infty \mathcal{A}^i \begin{bmatrix} W_h & 0 \\ 0 & 0 \end{bmatrix} (\mathcal{A}^T)^i \right) C_\theta^T \tag{52}$$

and by solving the discrete-time Lyapunov equation

$$\mathcal{A}X_h\mathcal{A}^T - X_h + \begin{bmatrix} W_h & 0 \\ 0 & 0 \end{bmatrix} = 0 \tag{53}$$

we readily have

$$F^2(\theta) = C_1 W_\theta C_1^T + C_\theta X_h C_\theta^T \tag{54}$$

Hence by (42), we immediately have the following result.

Theorem 1: The L_∞/L_2-induced norm $\|\mathcal{F}\|_{\infty/2}$ associated with the SISO LTI sampled-data system $\Sigma_{\rm SD}$ is given by

$$\|\mathcal{F}\|_{\infty/2} = \sup_{0\le\theta<h} \left(C_1 W_\theta C_1^T + C_\theta X_h C_\theta^T\right)^{1/2} \tag{55}$$

Even though Theorem 1 gives an almost direct method for the computation of the L_∞/L_2-induced norm $\|\mathcal{F}\|_{\infty/2}$ of SISO sampled-data systems, taking the supremum over $[0, h)$ precisely is bothersome. Regarding this issue, the following result for approximate computation follows readily.

Theorem 2: Let $M \in \mathbb{N}$ and $h' := h/M$. Then,

$$\max_{\theta\in\{0,h',\cdots,(M-1)h'\}} (C_1 W_\theta C_1^T + C_\theta X_h C_\theta)^{1/2} \to \|\mathcal{F}\|_{\infty/2} \tag{56}$$

as $M \to \infty$.

V. Conclusion

This paper tackled the problem of characterizing the induced norm from L_2 to L_∞ in single-input/single-output (SISO) LTI sampled-data systems. Behind the interest in this problem lied the two facts that (i) this induced norm coincides with the H_2 norm when we confine ourselves to SISO continuous-time LTI systems as a special class of systems under consideration, while (ii) there exist two conceptually different definitions for the H_2 norm of sampled-data systems [19]–[22]. We first gave a closed-form expression for the induced norm and argued that it coincides with neither of the two existing definitions for the H_2 norm. In particular, we showed that it is at least as large as (a more commonly used) one of the two definitions. We then gave a more sophisticated closed-form expression, by which we established an approximate but asymptotically exact method for computing the induced norm. These results are believed to shed a new light on the consequences of the hybrid and periodically time-varying nature of sampled-data systems. Finally, we would like to remark that the induced norm studied in this paper can be regarded as a new definition of the H_2 norm of sampled-data systems, and the optimal controller synthesis problem of minimizing the induced norm may be an interesting future topic.

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